RECURSIVE METHOD FOR EVALUATING CERTAIN VALUES OF SOME DIRICHLET SERIES VIA FOURIER SERIES OF POWER FUNCTIONS

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Abstract. This paper aims to present and justify a recursive method for calculating the values of the Riemann Zeta function, and Dirichlet Eta function, at positive even integers through the Fourier series expansion of $f_n(x) = x^{2n}$, and an analogous method for calculating the values of the Dirichlet Beta function at positive odd integers by expanding $g_n(x) = x^{2n-1}$. During the development of the work, it was observed that the Fourier coefficients of f_n and g_n, denoted respectively by $a_{k,n}$ and $b_{k,n}$, should be the solutions of Initial Value Problems (IVPs) of Difference Equations (DEs), which were solved intuitively and resulted in the recursive formulae that were applied in programs using the open-source mathematical computing software GNU Octave.

Keywords: Difference equations; Dirichlet beta function; Dirichlet eta function; Fourier coefficients; Riemann zeta function.
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Introduction

Definition 1 ([1]). *The Riemann Zeta function is given by*

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \Re(s) > 1.$$

Definition 2 ([1]). *The Dirichlet Eta function is given by*

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s}, \quad \Re(s) > 0.^1$$

Definition 3 ([2]). *The Dirichlet Beta function is given by*

$$\beta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^s}, \quad \Re(s) > 0.$$

These functions are classic examples of so-called Dirichlet series, whose general form is given by:

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[1]Although it is not called Dirichlet Eta function in that reference, it was decided to name her that way because it is the name by which she is best known.

$$\sum_{k=1}^{\infty} \frac{a_k}{k^s},$$

where s is a complex number, and a_k is a sequence defined in the field of complex numbers.

The interest in studying these series stems mainly from their close relationship with prime numbers through their factorization formulas as Euler products (see [1] and [2]):

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

where $\mathbb{P}$ is the set of positive prime numbers;

$$\beta(s) = \prod_{p \equiv 1 \mod 4} \frac{1}{1 - p^{-s}} \prod_{p \equiv 3 \mod 4} \frac{1}{1 + p^{-s}},$$

where $p \equiv 1 \mod 4$ are the primes of the form $4n - 3$ $(5, 13, 17, ...)$ and $p \equiv 3 \mod 4$ are the primes of the form $4n - 1$ $(3, 7, 11, ...)$.

Another definition of Dirichlet Eta function is given by the functional equation, relating with the Riemann Zeta function,

$$\eta(s) = (1 - 2^{1-s})\zeta(s).$$

Euler proved in [3], for $s = 2n$, with n a natural number, the following explicit formulae:

$$\zeta(2n) = \frac{B_{2n}(-1)^{n+1}2^{2n-1}\pi^{2n}}{(2n)!}, \tag{1}$$

where B_k is the k-th Bernoulli number; And for $s = 2n + 1$,

$$\beta(2n+1) = \frac{(-1)^n E_{2n}}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1}, \tag{2}$$

where E_k is the k-th Euler number. Our goal at the end will be to arrive at recursive expressions for equations (1) and (2) that do not contain the numbers E_{2n} or B_{2n} explicitly. For this, we will need a known definition and theorem from Fourier series theory, present in classical references [6], [7] and [8], which will be stated below.

Definition 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely integrable function of period $2L$, the Fourier series of f is defined as*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}\right),$$

in which

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx \quad \textit{and} \quad b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx$$

are the Fourier coefficients of f.

Theorem 5. *Let $f:\mathbb{R}\to\mathbb{R}$ be a piecewise differentiable function of period $2L$, then the Fourier series of f converges for all values of x to the average of the lateral limits of f at the point x, thus:*

$$\frac{f(x^+)+f(x^-)}{2}=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi x}{L}+b_k\sin\frac{k\pi x}{L}\right),$$

given that:

$$f(x^+)=\lim_{t\to x^+}f(t)\quad and\quad f(x^-)=\lim_{t\to x^-}f(t).$$

Although the references [4] and [5] present the results of this article. The importance of this work lies in the fact that it will present a considerably different way of evaluating specific values of these Dirichlet series, using, in addition to Fourier series theory, some theory of Difference Equations, as presented in the reference [9]. However, a very advanced prior knowledge of this subject is not considered necessary, as all results will be justified during the development process, although some prior contact is helpful.

Motivation

Before we get to the general expressions for calculating the Riemann Zeta function (and the Dirichlet Eta function) for positive even integers, and the Dirichlet Beta function for positive odd integers, we will begin by providing some well-known examples, present in classical literatures previously mentioned, of how to calculate the values of $\zeta(2),\eta(2)$ and $\beta(1)$ via periodic expansion of the functions $f(x)=x^2$ and $g(x)=x$ respectively.

Example 6. *Let $f:\mathbb{R}\to\mathbb{R}$ be periodic with period 2π defined as*

$$f(x)=x^2\ (-\pi\le x\le\pi).$$

Let's calculate the Fourier coefficients of f.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\,dx=\frac{2}{\pi}\left[\frac{x^3}{3}\right]\Bigg|_0^{\pi}=\frac{2\pi^2}{3}.$$

and using integration by parts,

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kx\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos kx\,dx$$

$$=\frac{2}{\pi}\left[\frac{x^2\sin kx}{k}\Bigg|_0^{\pi}-\int_0^{\pi}\frac{2x\sin kx}{k}\,dx\right]=-\frac{4}{k\pi}\left[\int_0^{\pi}x\sin kx\,dx\right]$$

$$=-\frac{4}{k\pi}\left[-\frac{x\cos kx}{k}\Bigg|_0^{\pi}+\int_0^{\pi}\frac{\cos kx}{k}\right]=-\frac{4}{k^2\pi}\left[-x\cos kx+\frac{\sin kx}{k}\right]\Bigg|_0^{\pi}=(-1)^k\frac{4}{k^2}.$$

Since f is an even function,

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin kx\,dx=0.$$

Therefore, the Fourier series of f is given by

$$\frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2}.$$

By Theorem 5, we know, because f is piecewise differentiable and continuous, that for all x belonging to $[-\pi, \pi]$, this series converges to $f(x) = x^2$, i.e.

$$x^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2}, \quad \forall x \in [-\pi, \pi]. \tag{3}$$

putting $x = \pi$ in (3),

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k \cos k\pi}{k^2},$$

given that for all $k \in \mathbb{N}$, $\cos k\pi = (-1)^k$ and $(-1)^{2k} = 1$, we have to

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2} \Rightarrow 4\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{2\pi^2}{3} \Rightarrow \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{4}$$

putting $x = \dfrac{\pi}{2}$ in (3),

$$\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos \frac{k\pi}{2},$$

given that

$$\cos \frac{k\pi}{2} = \begin{cases} 0, & \text{if } k \equiv 1 \mod 2, \\ 1, & \text{if } k \equiv 0 \mod 4, \\ -1, & \text{if } k \equiv 2 \mod 4, \end{cases}$$

we have to

$$\frac{\pi^2}{4} = \frac{\pi^2}{3} + 4\sum_{j=1}^{\infty} \frac{(-1)^j}{(2j)^2} \Rightarrow -\frac{\pi^2}{12} = \frac{4}{2^2}\sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \Rightarrow \eta(2) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} = \frac{\pi^2}{12}. \tag{5}$$

Example 7. *Let $g : \mathbb{R} \to \mathbb{R}$ be periodic with period 2π defined as*

$$g(x) = x \ (-\pi < x \leq \pi).$$

Let's calculate the Fourier coefficients of g, since g is a odd function,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\, dx = 0.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos kx\, dx = 0.$$

Again by integration by parts,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin kx\,dx$$

$$= \frac{2}{\pi}\left[\frac{-x}{k}\cos kx\Big|_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos kx\,dx\right] = \frac{2}{\pi}\left[\frac{-\pi\cos k\pi}{k}\right] = \frac{2(-1)^{k+1}}{k}.$$

The Fourier series of g is given by

$$2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin kx}{k}.$$

By Theorem 5, we know, because g is piecewise differentiable, that for all x belonging to $(-\pi,\pi)$, this series converges to $g(x)=x$, i.e.

$$x = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin kx}{k}.$$

putting $x=\dfrac{\pi}{2}$,

$$\frac{\pi}{2} = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\frac{k\pi}{2},$$

given that

$$\sin\frac{k\pi}{2} = \begin{cases} 0, & \text{if } k\equiv 0 \mod 2,\\ 1, & \text{if } k\equiv 1 \mod 4,\\ -1, & \text{if } k\equiv 3 \mod 4,\end{cases}$$

we have to

$$\beta(1) = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{(2j-1)} = \frac{\pi}{4}. \tag{6}$$

Generalization of method for even power functions

We will admit by convenience, for all sequences $(a_k)_{k\in\mathbb{N}}$,

$$\sum_{k=1}^{0} a_k \equiv 0.$$

Denote $a_{k,n}$ as the k-th Fourier coefficient of $f_n:\mathbb{R}\to\mathbb{R}$, periodic with period 2π, given by $f_n(x)=x^{2n}$ $(-\pi\le x\le\pi)$, therefore

$$a_{k,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2n}\cos kx\,dx.$$

[2]Notice that the Fourier coefficients accompanying $\sin kx$ are 0 for all natural numbers k and n, since f_n is always an even function.

For $k = 0$,

$$a_{0,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2n}\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^{2n}\,dx = \frac{2}{\pi}\cdot\frac{x^{2n+1}}{2n+1}\bigg|_0^{\pi} = \frac{2\pi^{2n}}{2n+1}. \tag{7}$$

For $k > 1$, by integration by parts, we see that

$$a_{k,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2n}\cos kx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^{2n}\cos kx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^{2n}\sin kx}{k}\bigg|_0^{\pi} - \frac{2n}{k}\int_0^{\pi} x^{2n-1}\sin kx\,dx\right] = \frac{4n}{k\pi}\int_0^{\pi} x^{2n-1}(-\sin kx)\,dx$$

$$= \frac{4n}{k\pi}\left[\frac{x^{2n-1}\cos kx}{k}\bigg|_0^{\pi} - \frac{2n-1}{k}\int_0^{\pi} x^{2n-2}\cos kx\,dx\right]$$

$$= \frac{4n}{k\pi}\left[\frac{\pi^{2n-1}(-1)^k}{k} - \frac{2n-1}{k}\int_0^{\pi} x^{2(n-1)}\cos kx\,dx\right]$$

$$\Rightarrow a_{k,n} = \frac{4n\pi^{2(n-1)}(-1)^k}{k^2} - \frac{(2n)(2n-1)a_{k,n-1}}{k^2}. \tag{8}$$

Thus, if k is fixed, the $a_{k,n}$ obey the first-order linear Difference Equation (DE) in n above (see [9]). In fact, if we take $n = 1$ in (8), we naturally have

$$a_{k,1} = \frac{4\pi^{2(1-1)}(-1)^k}{k^2} - \frac{2a_{k,0}}{k^2} = \frac{4(-1)^k}{k^2}$$

which is the result we had already found in Example 6, given that

$$a_{k,0} = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos kx = 0.$$

Now, since our process is independent of the chosen n, we can apply equation (8) to $a_{k,n-1}$,

$$a_{k,n-1} = \frac{4(n-1)\pi^{2(n-2)}(-1)^k}{k^2} - \frac{2(n-1)(2n-3)a_{k,n-2}}{k^2},$$

and substituting in (8), we have

$$a_{k,n} = \frac{4n\pi^{2(n-1)}(-1)^k}{k^2} - \frac{2n(2n-1)}{k^2}\left[\frac{4(n-1)\pi^{2(n-2)}(-1)^k}{k^2} - \frac{2(n-1)(2n-3)a_{k,n-2}}{k^2}\right]$$

$$\Rightarrow a_{k,n} = \frac{2(2n)!\pi^{2(n-1)}(-1)^k}{(2n-1)!k^2} - \frac{2(2n)!\pi^{2(n-2)}(-1)^k}{(2n-3)!k^4} + \frac{(2n)!a_{k,n-2}}{(2(n-2))!k^4}. \tag{9}$$

We could take $n = 2$ in equation (9) above to calculate $a_{k,2}$, however, since we want to find a more general expression, we will apply $a_{k,n-2}$ to equation (8) and substitute into (9).

$$a_{k,n-2} = \frac{4(n-2)\pi^{2(n-3)}(-1)^k}{k^2} - \frac{2(n-2)(2n-5)a_{k,n-3}}{k^2}$$

$$\Rightarrow a_{k,n} = \frac{2(2n)!\pi^{2(n-1)}(-1)^k}{(2n-1)!k^2} - \frac{2(2n)!\pi^{2(n-2)}(-1)^k}{(2n-3)!k^4}$$
$$+\frac{(2n)!}{(2(n-2))!k^4}\left[\frac{4(n-2)\pi^{2(n-3)}(-1)^k}{k^2} - \frac{2(n-2)(2n-5)a_{k,n-3}}{k^2}\right]$$

(10)
$$\Rightarrow a_{k,n} = \frac{2(2n)!\pi^{2(n-1)}(-1)^k}{(2n-1)!k^2} - \frac{2(2n)!\pi^{2(n-2)}(-1)^k}{(2n-3)!k^4} + \frac{2(2n)!\pi^{2(n-3)}(-1)^k}{(2n-5)!k^6} - \frac{(2n)!a_{k,n-3}}{(2(n-3))!k^6}.$$

Since $a_{k,0} = 0$, we can infer a solution for the equation (8) intuitively, and consequently for the equations (9) and (10), which will be presented and demonstrated in the following lemma

Lemma 8. *For all k and n natural numbers,*

$$a_{k,n} = 2(2n)!(-1)^k \sum_{s=1}^{n} \frac{(-1)^{s+1}\pi^{2(n-s)}}{(2(n-s)+1)!k^{2s}}. \tag{11}$$

Proof. For $n = 1$, we have $a_{k,1} = \dfrac{4(-1)^k}{k^2}$ which is the k-th Fourier coefficient of $f_1 \equiv f$ seen in the Example 6 of Motivation, now, we substitute (11) in (8)

$$\frac{2(2n)\pi^{2(n-1)}(-1)^k}{k^2} - \frac{(2n)(2n-1)}{k^2}\left[2(2(n-1))!(-1)^k \sum_{s=1}^{n-1} \frac{(-1)^{s+1}\pi^{2(n-1-s)}}{(2(n-1-s)+1)!k^{2s}}\right]$$
$$= \frac{2(2n)\pi^{2(n-1)}(-1)^k}{k^2} + 2(2n)!(-1)^k \sum_{s=1}^{n-1} \frac{(-1)^{s+2}\pi^{2(n-1-s)}}{(2(n-1-s)+1)!k^{2s+2}}$$
$$= \frac{2(2n)\pi^{2(n-1)}(-1)^k}{k^2} + 2(2n)!(-1)^k \sum_{t=2}^{n} \frac{(-1)^{t+1}\pi^{2(n-t)}}{(2(n-t)+1)!k^{2t}}$$[3]
$$= 2(2n)!(-1)^k \sum_{t=1}^{n} \frac{(-1)^{t+1}\pi^{2(n-t)}}{(2(n-t)+1)!k^{2t}} = a_{k,n}.$$

This shows us that expression (11) is a solution to equation (8), (9), and (10). We only need to prove that this solution is unique to be able to affirm that, in fact, expression (11) gives us the $k-$th Fourier coefficient of f_n.

Based on reference [9]. Assuming that there exists another different solution $\alpha_{k,n}$ for equation (8), which satisfies the same initial condition $\alpha_{k,1} = 4(-1)^k/k^2$. Let $S = \{n \in \mathbb{N} \mid a_{k,n} \neq \alpha_{k,n}\}$, by the hypothesis that the solutions are different, $S \neq \varnothing$, by the Well-Ordering Principle, we know that S has a smallest element, which we will denote by n_0, it is known that $n_0 \geq 2$ and we can see that

$$a_{k,n_0} = \frac{2}{k^2}\left[(2n_0)\pi^{2(n_0-1)}(-1)^k - (2n_0)(2n_0-1)\frac{a_{k,n_0-1}}{2}\right]$$
$$= \frac{2}{k^2}\left[(2n_0)\pi^{2(n_0-1)}(-1)^k - (2n_0)(2n_0-1)\frac{\alpha_{k,n_0-1}}{2}\right] = \alpha_{k,n_0},$$

[3]Doing the change of variable $t = s + 1$.

which is a contradiction, because $n_0 \in S$, and proves that

$$a_{k,n} = \alpha_{k,n},\ \forall k, n \in \mathbb{N}.$$

□

Theorem 9. *For all n natural,*

$$\zeta(2n) = \frac{(-1)^{n+1} n \pi^{2n}}{(2n+1)!} + \sum_{s=1}^{n-1} \frac{(-1)^{n+s+1} \pi^{2(n-s)} \zeta(2s)}{(2(n-s)+1)!},$$

$$\eta(2n) = \frac{(-1)^{n+1} \pi^{2n} \left(2^{2n} - (2n+1)\right)}{2(2n+1)!} + \sum_{s=1}^{n-1} \frac{(-1)^{n+s+1} (2\pi)^{2(n-s)} \eta(2s)}{(2(n-s)+1)!}.$$

Proof. By Lemma 8, Theorem 5 and equation (7), we know, because f_n is piecewise differentiable and continuous, that for all x belonging to $[-\pi, \pi]$, the fourier series of f_n converges to $f_n(x) = x^{2n}$, i.e.

$$x^{2n} = \frac{\pi^{2n}}{2n+1} + \sum_{k=1}^{\infty} 2(2n)!(-1)^k \cos kx \sum_{s=1}^{n} \frac{(-1)^{s+1} \pi^{2(n-s)}}{(2(n-s)+1)! k^{2s}}. \tag{12}$$

Putting $x = \pi$ in (12),

$$\pi^{2n} = \frac{\pi^{2n}}{2n+1} + 2(2n)! \sum_{s=1}^{n} \frac{(-1)^{s+1} \pi^{2(n-s)}}{(2(n-s)+1)!} \sum_{k=1}^{\infty} \frac{1}{k^{2s}}.$$

$$\Rightarrow \frac{n}{(2n+1)!} = \sum_{s=1}^{n} \frac{(-1)^{s+1} \pi^{-2s} \zeta(2s)}{(2(n-s)+1)!}. \tag{13}$$

By performing some algebraic manipulations in (13), we can give $\zeta(2n)$ explicitly, which will be useful for applying the method to some values of n.

$$\zeta(2n) = \frac{(-1)^{n+1} n \pi^{2n}}{(2n+1)!} + \sum_{s=1}^{n-1} \frac{(-1)^{n+s+1} \pi^{2(n-s)} \zeta(2s)}{(2(n-s)+1)!}. \tag{14}$$

Putting $x = \dfrac{\pi}{2}$ in (12),

$$\left(\frac{\pi}{2}\right)^{2n} = \frac{\pi^{2n}}{2n+1} + 2(2n)! \sum_{s=1}^{n} \frac{(-1)^{s+1} \pi^{2(n-s)}}{(2(n-s)+1)!} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)^{2s}}.$$

$$\Rightarrow \frac{(2n+1) - 2^{2n}}{2^{2n+1}(2n+1)!} = \sum_{s=1}^{n} \frac{(-1)^{s+1} \pi^{-2s}}{2^{2s}(2(n-s)+1)!} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2s}}.$$

$$\Rightarrow \frac{2^{2n} - (2n+1)}{2^{2n+1}(2n+1)!} = \sum_{s=1}^{n} \frac{(-1)^{s+1} (2\pi)^{-2s} \eta(2s)}{(2(n-s)+1)!}. \tag{15}$$

By manipulating (15) just as we did for the equation (14), we have the explicit formula

$$\eta(2n) = \frac{(-1)^{n+1}\pi^{2n}\left(2^{2n} - (2n+1)\right)}{2(2n+1)!} + \sum_{s=1}^{n-1} \frac{(-1)^{n+s+1}(2\pi)^{2(n-s)}\eta(2s)}{(2(n-s)+1)!} \tag{16}$$

□

We can now apply some values of n to the equations of the Theorem 9, to calculate some values of $\zeta(2n)$ and $\eta(2n)$).
For $n = 1$:

$$\zeta(2) = \frac{(-1)^2 \cdot 1 \cdot \pi^2}{3!} = \frac{\pi^2}{6};$$

$$\eta(2) = \frac{(-1)^2 \cdot 1 \cdot \pi^2}{2 \cdot 3!} = \frac{\pi^2}{12}.$$

For $n = 2$:

$$\zeta(4) = \frac{(-1) \cdot 2 \cdot \pi^4}{5!} + \frac{\pi^2\zeta(2)}{3!} = \frac{\pi^4}{36} - \frac{\pi^4}{60} = \frac{5\pi^4 - 3\pi^4}{180} = \frac{\pi^4}{90};$$

$$\eta(4) = \frac{(-1) \cdot 11 \cdot \pi^4}{2 \cdot 5!} + \frac{(2\pi)^2\eta(2)}{3!} = \frac{\pi^4}{18} - \frac{11\pi^4}{240} = \frac{(40/3 - 11)\pi^4}{240} = \frac{7\pi^4}{720}.$$

For $n = 3$:

$$\zeta(6) = \frac{3\pi^6}{7!} - \frac{\pi^4\zeta(2)}{5!} + \frac{\pi^2\zeta(4)}{3!} = \frac{\pi^6}{1680} - \frac{\pi^6}{720} + \frac{\pi^6}{540} = \frac{28\pi^6/9 - 7\pi^6/3 + \pi^6}{1680}$$

$$= \frac{56\pi^6 - 24\pi^6}{18 \cdot 1680} = \frac{32\pi^6}{9 \cdot 2^5 \cdot 105} = \frac{\pi^6}{945},$$

$$\eta(6) = \frac{57\pi^6}{2 \cdot 7!} - \frac{(2\pi)^4\eta(2)}{5!} + \frac{(2\pi)^2\eta(4)}{3!} = \frac{(57 - 112 + 196/3)\pi^6}{2 \cdot 7!} = \frac{171 - 336 + 196}{3 \cdot 2 \cdot 7!} = \frac{31\pi^6}{30240}.$$

We quickly realized that calculations for larger values of n become increasingly complex. Therefore, using the GNU Octave software [10], programs were created that calculates, with some precision, the values of the $\zeta(2k)$ and $\eta(2k)$, for $k \leq n$, with the method described above.

Let's show the codes for Riemann Zeta function and Dirichlet Eta function, and the results for $n = 20$ and $n = 10$ respectively. Larger values of n cause glaring inaccuracies due to technical limitations of the software itself.

```
clc;
clear all;
format long;

% Variables
n  = input('Enter the nth even number you want to calculate the zeta
    values for:');
fprintf('\n');
k = 1
K = k;
fprintf('\n');
zeta = ((-1)^(k+1))*k*pi^(2*k)/factorial(2*k+1)
Z = zeta;
```

```
fprintf('\n');
k = k+1
fprintf('\n');

% Recurrence
while k <= n
  s = 1;
  a =  (-1)^(s+k+1)*pi^(2*(k-s))/factorial(2*(k-s)+1);
  A = a;
while s < k - 1
  s = s+1;
  a =  (-1)^(s+k+1)*pi^(2*(k-s))/factorial(2*(k-s)+1);
  A = [A,a];
endwhile
  zeta = ((-1)^(k+1))*k*pi^(2*k)/factorial(2*k+1) + dot(A,Z)
  fprintf('\n');
  Z = [Z,zeta];
  fprintf('\n');
  K = [K,k];
  k = k+1
endwhile

% Ploting the results
graphics_toolkit("qt")
plot(K, Z, 'b-', 'LineWidth', 1.5);
title('Riemann Zeta Function at Positive Even Integers');
xlabel('k');
ylabel('\zeta(2k)');
```

```
Enter the nth even number you want to calculate the zeta values for:20

k = 1

z = 1.644934066848226

k = 2

z = 1.082323233711138


k = 3
z = 1.017343061984449


k = 4
z = 1.004077356197945


k = 5
z = 1.000994575127819

```

```
23
24 k = 6
25 z = 1.000246086553308
26
27
28 k = 7
29 z = 1.000061248135059
30
31
32 k = 8
33 z = 1.000015282259410
34
35 k = 9
36 z = 1.000003817293267
37
38
39 k = 10
40 z = 1.000000953962036
41
42
43 k = 11
44 z = 1.000000238450505
45
46
47 k = 12
48 z = 1.000000059608191
49
50
51 k = 13
52 z = 1.000000014901557
53
54
55 k = 14
56 z = 1.000000003725336
57
58
59 k = 15
60 z = 1.000000000931330
61
62
63 k = 16
64 z = 1.000000000232834
65
66
67 k = 17
68 z = 1.000000000058210
69
70
71 k = 18
72 z = 1.000000000014555
73
```

```
74
75 k = 19
76 z = 1.000000000003641
77
78
79 k = 20
80 z = 1.000000000000913
```

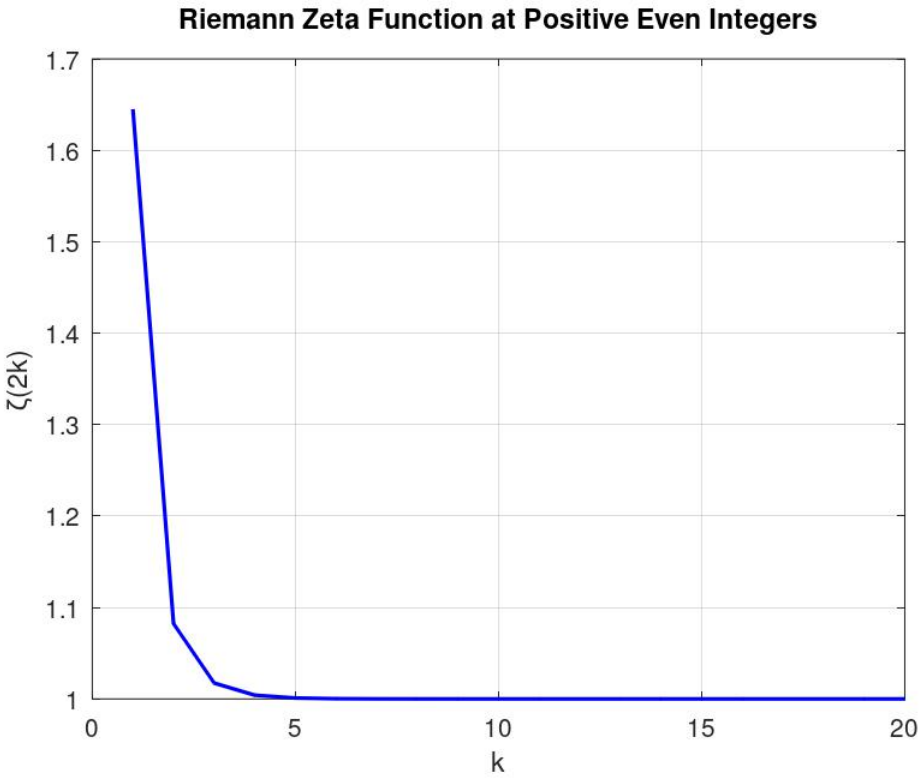

Figure 1. Riemann Zeta values at the first 20 even numbers via GNU Octave.

```
1  clc;
2  clear all;
3  format long;
4
5  % Variables
6  n = input('Enter the nth even number you want to calculate the eta
       values for:');
7  fprintf('\n');
8  k = 1
9  K = k;
10 fprintf('\n');
11 eta = (-1)^(k+1)*(pi^(2*k))*(2^(2*k)-(2*k+1))/(2*factorial(2*k+1))
12 E = eta;
13 fprintf('\n');
```

```
k = k+1
fprintf('\n');

% Recurrence
  while k <= n
  s = 1;
  a =  (-1)^(s+k+1)*((2*pi)^(2*(k-s)))/factorial(2*(k-s)+1);
  A = a;
  while s < k - 1
  s = s+1;
  a =  (-1)^(s+k+1)*((2*pi)^(2*(k-s)))/factorial(2*(k-s)+1);
  A = [A,a];
endwhile
  eta = (-1)^(k+1)*(pi^(2*k))*(2^(2*k)-(2*k+1))/(2*factorial(2*k+1))+
      dot(A,E)
  fprintf('\n');
  E = [E,eta];
  fprintf('\n');
  K = [K,k];
  k = k+1
endwhile

% Ploting the results
graphics_toolkit("qt")
plot(K, E, 'b-', 'LineWidth', 1.5);;
title('Dirichlet Eta Function at Positive Even Integers');
xlabel('k');
ylabel('\eta(2k)');
```

```
Enter the nth even number you want to calculate the eta values for:10

k = 1

eta = 0.822467033424113

k = 2

eta = 0.947032829497246


k = 3
eta = 0.985551091297436


k = 4
eta = 0.996233001852656


k = 5
eta = 0.999039507598314

```

```
k = 6
eta = 0.999757685144052


k = 7
eta = 0.999939170346794


k = 8
eta = 0.999984764218220


k = 9
eta = 0.999996187882938


k = 10
eta = 0.999999046664977
```

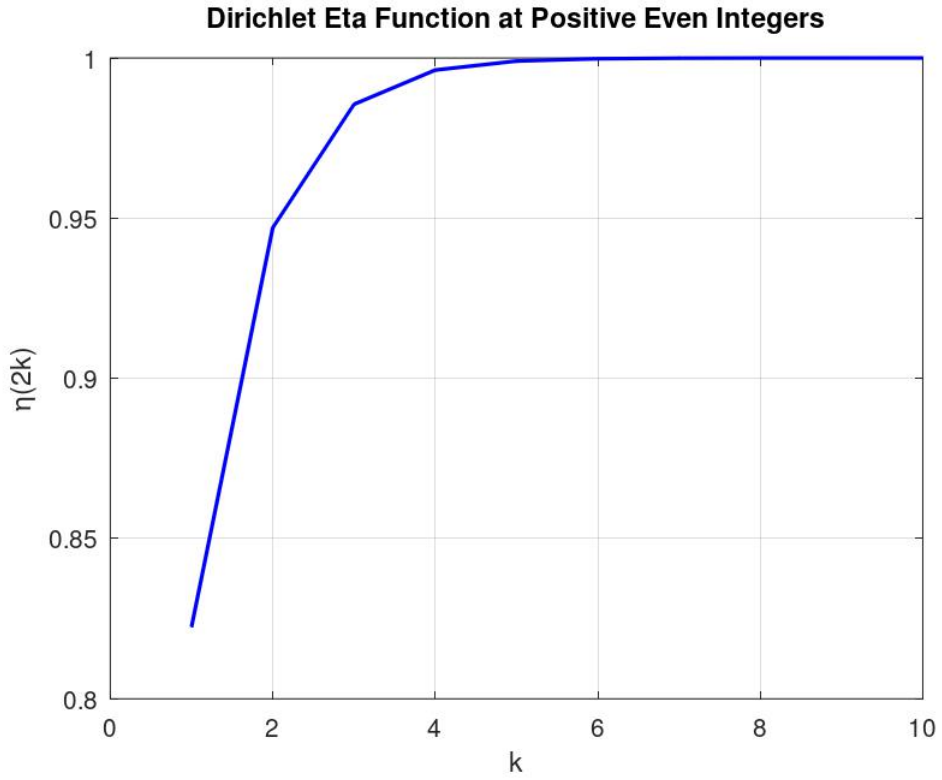

Figure 2. Dirichlet Eta values at the first 10 even numbers via GNU Octave.

Generalization of method for odd power functions

Let $g_n : \mathbb{R} \to \mathbb{R}$ of period 2π defined for all $n \in \mathbb{N}$ as

$$g_n(x) = x^{2n-1}(-\pi < x \leq \pi).$$

We will denote $b_{k,n}$ as the k-th Fourier coefficient of g_n, so

$$b_{k,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2n-1}\sin kx\,dx.^4$$

Again, we will apply the integration by parts technique to arrive at another DE,

$$\begin{aligned} b_{k,n} &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2n-1}\sin kx\,dx. = \frac{2}{\pi}\int_{0}^{\pi} x^{2n-1}\sin kx\,dx \\ &= \frac{2}{\pi}\left[-\frac{x^{2n-1}\cos kx}{k}\bigg|_0^{\pi} + \frac{(2n-1)}{k}\int_0^{\pi} x^{2n-2}\cos kx\,dx\right] \\ &= \frac{2\pi^{2(n-1)}(-1)^{k+1}}{k} + \frac{2(2n-1)}{k\pi}\int_0^{\pi} x^{2(n-1)}\cos kx\,dx. \end{aligned}$$

Notice that if $n = 1$, the integral of the equation above is zero, therefore

$$b_{k,1} = \frac{2(-1)^{k+1}}{k}.$$

which is the result we had already found in Example 7 of Motivation, let's calculate the integral above for arbitrary n

$$\begin{aligned} \int_0^{\pi} x^{2(n-1)}\cos kx\,dx &= \frac{x^{2(n-1)}\sin kx}{k}\bigg|_0^{\pi} - \frac{2(n-1)}{k}\int_0^{\pi} x^{2(n-1)-1}\sin kx\,dx \\ &= -\frac{(n-1)\pi b_{k,n-1}}{k}, \end{aligned}$$

which leads us to the following first order linear DE in n,

$$b_{k,n} = \frac{2\pi^{2(n-1)}(-1)^{k+1}}{k} - \frac{(2n-1)2(n-1)b_{k,n-1}}{k^2}. \tag{17}$$

Analogously to the previous case, since n is arbitrary, we can replace n with $n-1$ in the equation above.

$$b_{k,n-1} = \frac{2\pi^{2(n-2)}(-1)^{k+1}}{k} - \frac{(2n-3)2(n-2)b_{k,n-2}}{k^2},$$

and replacing in (17),

$$b_{k,n} = \frac{2\pi^{2(n-1)}(-1)^{k+1}}{k} - \frac{2\pi^{2(n-2)}(2n-1)!(-1)^{k+1}}{(2n-3)!k^3} + \frac{(2n-1)!b_{k,n-2}}{(2n-5)!k^4}. \tag{18}$$

[4] Notice that, similarly to the previous case, the Fourier coefficients accompanying $\cos kx$ are zero, since g_n is always an odd function.

Continuing this process, similarly to the previous case, we can intuit a solution to this DE, which will be presented in the following lemma.

Lemma 10. *For all k and n natural numbers,*

$$b_{k,n} = 2(2n-1)!(-1)^k \sum_{s=1}^{n} \frac{(-1)^s \pi^{2(n-s)}}{(2(n-s)+1)! k^{2s-1}}. \tag{19}$$

Proof. For $n = 1$, it is verified that $b_{k,1} = \dfrac{2(-1)^{k+1}}{k}$, thus, satisfying our initial condition, let us show, that equation (19) is indeed a solution to our problem,

$$b_{k,n} = \frac{2\pi^{2(n-1)}(-1)^{k+1}}{k} - \frac{(2n-1)2(n-1)b_{k,n-1}}{k^2}.$$

$$= \frac{2\pi^{2(n-1)}(-1)^{k+1}}{k} - \frac{(2n-1)2(n-1)2(2n-3)!(-1)^k}{k^2} \sum_{s=1}^{n-1} \frac{(-1)^s \pi^{2(n-1-s)}}{(2(n-1-s)+1)! k^{2s-1}}$$

$$= \frac{2\pi^{2(n-1)}(-1)^{k+1}}{k} + 2(2n-1)!(-1)^k \sum_{s=1}^{n-1} \frac{(-1)^{s+1} \pi^{2(n-1-s)}}{(2(n-1-s)+1)! k^{2(s+1)-1}}$$

$$= \frac{2\pi^{2(n-1)}(-1)^{k+1}}{k} + 2(2n-1)!(-1)^k \sum_{t=2}^{n} \frac{(-1)^t \pi^{2(n-t)}}{(2(n-t)+1)! k^{2t-1}}.^{5}$$

$$= 2(2n-1)!(-1)^k \sum_{t=1}^{n} \frac{(-1)^t \pi^{2(n-t)}}{(2(n-t)+1)! k^{2t-1}} = b_{k,n}.$$

The uniqueness demonstration is done exactly as in the previous case, thus, we have that in fact, equation (19) gives us the k-th Fourier coefficient of g_n. □

Theorem 11. *For all n natural,*

$$\beta(2n-1) = \frac{(-1)^{n+1} 2^{-2n} \pi^{2n-1}}{(2n-1)!} + \sum_{s=1}^{n-1} \frac{(-1)^{n+s+1} \pi^{2(n-s)} \beta(2s-1)}{(2(n-s)+1)!} \tag{20}$$

Proof. By Theorem 5, we know that for $x \in (-\pi, \pi)$

$$g_n(x) = x^{2n-1} = \sum_{k=1}^{\infty} b_{k,n} \sin kx$$

putting $x = \dfrac{\pi}{2}$ and substituting the expression for $b_{k,n}$

$$\left(\frac{\pi}{2}\right)^{2n-1} = \sum_{k=1}^{\infty} 2(2n-1)!(-1)^k \sin\frac{k\pi}{2} \sum_{s=1}^{n} \frac{(-1)^s \pi^{2(n-s)}}{(2(n-s)+1)! k^{2s-1}}$$

$$\Rightarrow \frac{1}{2^{2n}(2n-1)!} = \sum_{s=1}^{n} \frac{(-1)^s \pi^{-2s+1}}{(2(n-s)+1)!} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2s-1}} \sin\frac{k\pi}{2}$$

[5] Again, making the change of variable $t = s + 1$.

$$\Rightarrow \frac{2^{-2n}}{(2n-1)!} = \sum_{s=1}^{n} \frac{(-1)^{s+1}\pi^{-(2s-1)}}{(2(n-s)+1)!} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j-1)^{2s-1}}$$

$$\Rightarrow \frac{2^{-2n}}{(2n-1)!} = \sum_{s=1}^{n} \frac{(-1)^{s+1}\pi^{-(2s-1)}\beta(2s-1)}{(2(n-s)+1)!} \tag{21}$$

As in the previous case, we can manipulate the equation above to explicitly express $\beta(2n-1)$]

$$\beta(2n-1) = \frac{(-1)^{n+1}2^{-2n}\pi^{2n-1}}{(2n-1)!} + \sum_{s=1}^{n-1} \frac{(-1)^{n+s+1}\pi^{2(n-s)}\beta(2s-1)}{(2(n-s)+1)!}.$$

□

For $n = 1$

$$\beta(1) = \frac{(-1)^2 2^{-2}\pi^1}{1!} = \frac{\pi}{4};$$

For $n = 2$

$$\beta(3) = \frac{(-1)^{2+1}2^{-4}\pi^3}{3!} + \frac{(-1)^4\pi^2\beta(1)}{3!} = \frac{\pi^3}{24} - \frac{\pi^3}{96} = \frac{4\pi^3 - \pi^3}{96} = \frac{3\pi^3}{96} = \frac{\pi^3}{32}.$$

Just as we did for even power functions, here is an GNU Octave program to calculate values of the Dirichlet Beta function, and its results for the first 15 odd numbers. Again, this number is arbitrary and was chosen due to software limitations.

```
  clc;
clear all;
format long;

% Variables
n  = input('Enter the nth odd number you want to calculate the beta
    values for:');
fprintf('\n');
k = 1
K = k;
fprintf('\n');
beta = ((-1)^(k+1))*2^(-2*k)*pi^(2*k-1)/factorial(2*k-1)
B = beta;
fprintf('\n');
k = k+1
fprintf('\n');

% Recurrence
while k <= n
  s = 1;
  a =  (-1)^(s+k+1)*pi^(2*(k-s))/factorial(2*(k-s)+1);
  A = a;
while s < k - 1
  s = s+1;
  a =  (-1)^(s+k+1)*pi^(2*(k-s))/factorial(2*(k-s)+1);
```

```
  A = [A,a];
endwhile
  beta = ((-1)^(k+1))*2^(-2*k)*pi^(2*k-1)/factorial(2*k-1) + dot(A,B)
  fprintf('\n');
  B = [B,beta];
  fprintf('\n');
  K = [K,k];
  k = k+1
endwhile

% Ploting the results
graphics_toolkit("qt")
plot(K, B, 'b-', 'LineWidth', 1.5);

title('Dirichlet Beta Function at Positive Odd Integers');
xlabel('k');
ylabel('\beta(2k-1)');
```

```
Enter the nth odd number you want to calculate the beta values for:15

k = 1

beta = 0.785398163397448

k = 2

beta = 0.968946146259369


k = 3
beta = 0.996157828077088


k = 4
beta = 0.999554507890540


k = 5
beta = 0.999949684187220


k = 6
beta = 0.999994374973823


k = 7
beta = 0.999999373583772


k = 8
beta = 0.999999930340843
```

```
34
35
36  k = 9
37  beta = 0.999999992257783
38
39
40  k = 10
41  beta = 0.999999999139662
42
43
44  k = 11
45  beta = 0.999999999904404
46
47
48  k = 12
49  beta = 0.999999999989379
50
51
52  k = 13
53  beta = 0.999999999998821
54
55
56  k = 14
57  beta = 0.999999999999871
58
59
60  k = 15
61  beta = 0.999999999999987
```

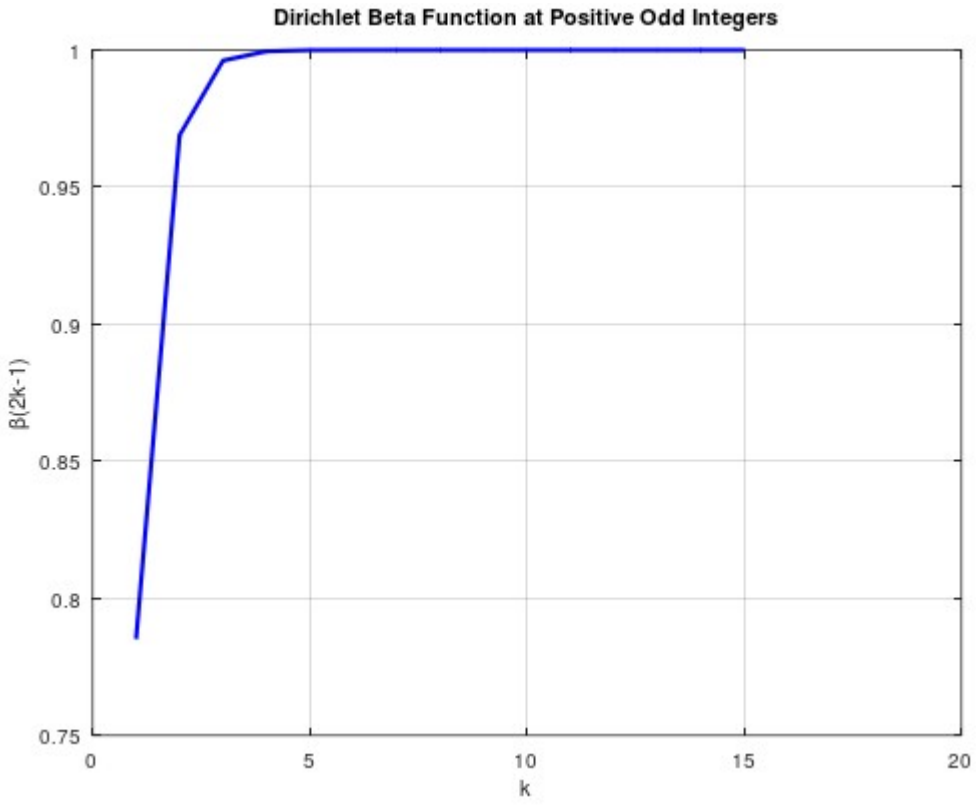

Figure 3. Dirichlet Beta values at the first 15 odd numbers via GNU Octave.

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